Linear Algebra

$x^2 = 4$

$x = \pm 2$

$x^2 = -4$ Let $i = \sqrt{-1}$

$x = \pm 2i$

Complex number

$a + ib$ $a, b \in \mathbb{R}$ separating real & imaginary parts

to model quantum computing states

$(2+3i)(4-8i) = 8 - 4i - 24i^2 = 32 - 4i$

Complex conjugate: negate imaginary part

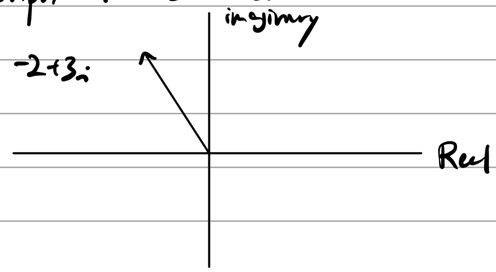
$(a+ib)^* = a - ib$

$(-2+3i)^* = -2-3i$

multiplying by complex conjugate $\rightarrow$ always $\mathbb{R}$

$(2+3i)(2-3i) = 4 - 9i^2 = 13$

Complex # as vector

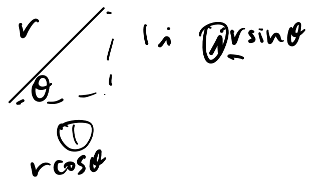


→ magnitude

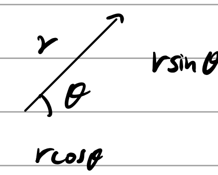
$|-2+3i| = \sqrt{4+9} = \sqrt{13}$

$|a+ib| = \sqrt{a^2+b^2}$

$|4-3i| = \sqrt{16+9} = 5$



represent with magnitude & angle with x-axis



$r = \sqrt{1^2+1^2} = \sqrt{2}$

$a+ib = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

$\tan\theta = \frac{b}{a}$

$\theta = \arctan\left(\frac{b}{a}\right)$

$a+bi = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

ex) $1+i$ into polar form:

$\tan\theta = 1$

$\arctan(1) = \theta = \frac{\pi}{4}$

$= re^{i\theta}$

$= \sqrt{2}\,e^{i\frac{\pi}{4}}$

$\tan\theta = 1 \qquad \theta = \arctan(1) = \frac{\pi}{4}$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

into exponential form: $\quad re^{i\theta}$

$1+i = \sqrt{2}\,e^{i\frac{\pi}{4}}$

quantum computing → exclusively use exponential form

change θ → rotate around circle.

$2\sqrt{2}\,e^{i\frac{\pi}{4}}$. $\quad 2\sqrt{2}\,e^{i\frac{3\pi}{4}}$. $\quad 2\sqrt{2}\,e^{i\,-\frac{\pi}{2}}$

multiplying to complex numbers in exponential form
↳ rotating around circle.
angles add together.
↳ useful in representing quantum states

$$e^{i\frac{\pi}{5}} e^{i\frac{\pi}{3}} = e^{i\frac{8\pi}{15}}$$

Matricies

$m \times n$ matrix 4×1 3×2

addition → same dimensions needed

multiplying by scalar.

Vector & Matrix Multiplication to Transform a Vector

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 5 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 3+0 & 5+0 \\ 4-5 & 6-2 & 10-0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3+8 \\ 6-4 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$

$AB \neq BA$

$n \times 1$ matrix "Column Vector"

graphed like any other vector

to represent a state of a quantum computation

multiplying a matrix with a column vector

→ another column vector

column vector getting "transformed" by the matrix

applying matricies to a quantum state to apply operations

on quantum computers

Identity Matrix (I)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad IA = A$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Inverse Matrix $A^{-1}A = I$

↳ column vector stays in the same spot

Unitary & Hermitian Matricies

$$A = \begin{bmatrix} 2+3i & 0 \\ 5 & 3-i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 2-3i & 0 \\ 5 & 3+i \end{bmatrix}$$

ex) $$\begin{bmatrix} 3+7i & 2e^{i\frac{\pi}{3}} \\ 5 & 3-i \end{bmatrix}^*$$

$$(re^{i\theta})^* = re^{-i\theta}$$

$$= \begin{bmatrix} 3-7i & 2e^{-i\frac{\pi}{3}} \\ 5 & 3+i \end{bmatrix}$$

Transpose of Matrix : exchanging rows for columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Dagger : transpose of conjugate / conjugate of transpose.

$$(A^*)^T = (A^T)^* = A^\dagger$$

Two types of matricies to apply operations.

1) unitary matrix, U

$$U^\dagger U = I \rightarrow U^\dagger = U^{-1}$$

unitary matrix acts on a vector

↳ length of vector stays the same. rotated or flipped.

Hermitian Matrix	$H = H^{\dagger}$

Eigenvectors & Eigenvalues

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 0.5+1.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

same direction but stretched or compressed
↳ factor out a scaler & get the same vector

Eigen vector of the transformation matrix

$$A\vec{v} = \lambda\vec{v}$$

Eigenvector Eigenvalue

ex) $$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= 2\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

eigenvalue: 2
eigenvector: $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Qubit & Superposition

Classical computers
↳ use bits . binary (0's & 1's)
to store & process data

Quantum computers
↳ use qubits (0 and 1 at the same time)

Qubits (Quantum bits)

↳ can be made from any quantum particle that has 2 distinct states

ex) photon being polarized either vertically / horizontally

defined with column vectors

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Superposition:

quantum particle is in two states simultaneously

ex) photon is both horizontally & vertically polarized

qubit is in superposition if both $|0\rangle$ & $|1\rangle$

Representing qubits mathematically,

column vector

how much the qubit is in the $|0\rangle$ state

$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

how much the qubit is in the $|1\rangle$ state

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ → all in the 0 state. none in 1

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

else → superposition of 0 & 1

When we mesure a quantum system, it collapses into the measured state.

anology: photon that is in superposition of both vertically & horizontally polarized.
mesure → we can only measure it as horizontally or vertically polarized.
once it has been measured, it will collapse into the mesured state.

we mesure it to be horizontally polarized, it becomes horizontally polarized

When we measure a qubit we can only measure a 0 or 1.

if we were to measure the qubit $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$,
we would not measure α or β.
we would still measure 0 or 1.

if measured $|\psi\rangle$ as 0 → $|\psi\rangle$ collapses into 0 state
→ $|\psi\rangle \Rightarrow |0\rangle$
if measured $|\psi\rangle$ as 1 → $|\psi\rangle$ collapses into 1 state
→ $|\psi\rangle \Rightarrow |1\rangle$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

probability of measuring a 0 or 1.

probability of measuring $|\psi\rangle$ as 0: $|\alpha|^2$

probability of measuring $|\psi\rangle$ as 1: $|\beta|^2$

ex) $|\psi_1\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$

probability of measuring 0: $\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$

probability of measuring 1: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ probability of measuring 0 is 1.

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ probability of measuring 1 is 1.

$|\alpha|^2 + |\beta|^2 = 1$ valid qubit stete.

$|\alpha|^2 + |\beta|^2 \neq 1$ $\rightarrow$ not valid qubit stete.

measure something $\rightarrow$ permanently changes the state of the system to the measurment

Dirac Notation

$$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \beta\end{pmatrix}$$

$$= \alpha\begin{pmatrix}1\\ 0\end{pmatrix} + \beta\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \alpha|0\rangle + \beta|1\rangle$$

$|x\rangle$: Ket vectors. represent quantum state.

ex) $$|\psi\rangle = \begin{pmatrix}\frac{1}{2}\\ \frac{2\sqrt{3}}{4}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \frac{2\sqrt{3}}{4}\end{pmatrix}$$

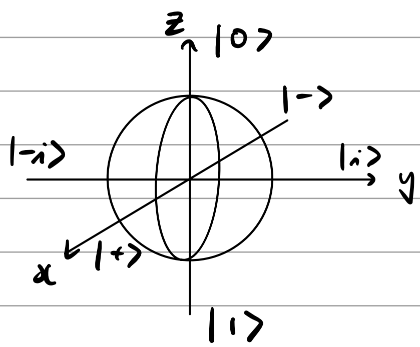
$$= \frac{1}{2}\begin{pmatrix}1\\ 0\end{pmatrix} + \frac{2\sqrt{3}}{4}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \frac{1}{2}|0\rangle + \frac{2\sqrt{3}}{4}|1\rangle$$

conventional way of writing a quantum computing state

Representing Qubit on the Bloch Sphere

Bloch sphere



higher vertically → higher probability of measuring $|\psi\rangle$ as $|0\rangle$

lower vertically → higher probability of measuring $|\psi\rangle$ as $|1\rangle$

$$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

↳ halfway between north & south poles.
even chance of being measured 0 and 1

qubit can spin around the sphere.

ex) $$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Manipulating a qubit: X. Y. Z Gates

gates to change the state of qubits

X gate $\quad |\psi\rangle = |0\rangle$

$\rightarrow |\psi\rangle = |1\rangle$

X-gate flips the qubit π radians around the x-axis on the Bloch sphere

Y gate:
flips the qubit π radians around the y-axis on the Bloch sphere

Z gate:
around z-axis

apply same gate twice → rotate around 2π radians

↳ end up in the original state.

X, Y, Z gates are their own inverses

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Applying X gate to arbitrary qubit $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$

$X|0\rangle \stackrel{?}{=} |1\rangle$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$U|0\rangle = \begin{pmatrix} a \\ c \end{pmatrix} = a|0\rangle + c|1\rangle$

$U|1\rangle = \begin{pmatrix} b \\ d \end{pmatrix} = b|0\rangle + d|1\rangle$

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$U|\psi\rangle = U(\alpha|0\rangle + \beta|1\rangle)$

$U|\psi\rangle = \alpha U|0\rangle + \beta U|1\rangle$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|\psi\rangle = Y\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

$$= \frac{\sqrt{3}}{2} Y|0\rangle + \frac{1}{2} Y|1\rangle$$

$$= \frac{\sqrt{3}}{2}\begin{pmatrix} 0 \\ i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -i \\ 0 \end{pmatrix}$$

$$= \frac{\sqrt{3}}{2} i \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2} i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\sqrt{3}}{2} i\, |1\rangle - \frac{1}{2} i\, |0\rangle$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z\left(\alpha|0\rangle + \beta|1\rangle\right) \stackrel{?}{=} \alpha|0\rangle - \beta|1\rangle$$

$$= \alpha Z|0\rangle + \beta Z|1\rangle$$

$$= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \alpha|0\rangle - \beta|1\rangle$$

Global & Relative Phase

rotate around z-axis "phase"

↳ probability of measuring 0 and 1 remains the same

representing phase

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \xrightarrow{Z} \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$Z\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

qubit rotated π radians around z-axis

$(-1 = e^{i\pi})$ → angle of complex number is π rad

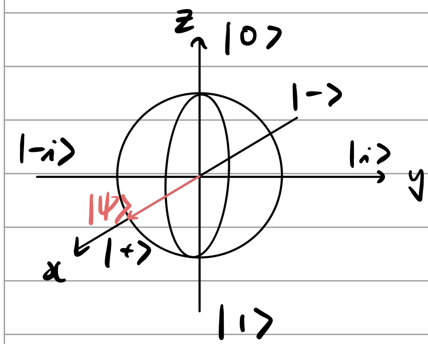
$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\pi}\frac{1}{\sqrt{2}}|1\rangle$

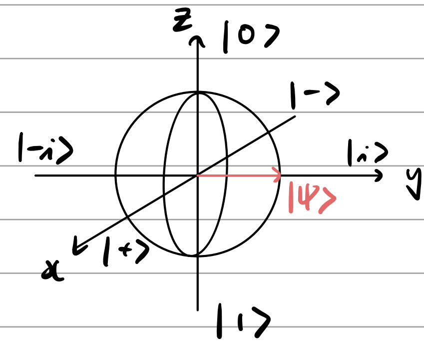
$\frac{1}{\sqrt{2}}|0\rangle + i\frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\frac{\pi}{2}}\frac{1}{\sqrt{2}}|1\rangle$

$(i = e^{i\frac{\pi}{2}})$

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$$\frac{1}{\sqrt{2}}|0\rangle + i\frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\frac{\pi}{2}}\frac{1}{\sqrt{2}}|1\rangle$$





rotate $\frac{\pi}{2}$ rad around z-axis

$e^{i\varphi}$, nice mathematical way of rotating around a circle by changing the value of φ

multiplying the $|1\rangle$ state of the qubit with the complex number $e^{i\varphi}$
↳ rotate the qubit around z-axis by φ rad.

Global phase → entire qubit is multiplied by complex number

$$e^{i\varphi}(\alpha|0\rangle + \beta|1\rangle)$$

$$= e^{i\varphi}\alpha|0\rangle + e^{i\varphi}\beta|1\rangle \equiv \alpha|0\rangle + \beta|1\rangle$$

discard.

Relative phase → just the $|1\rangle$ state multiplied by complex number

$$\alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

$e^{i\theta}\alpha|0\rangle + e^{i\varphi}\beta|1\rangle$

$= e^{i\theta}\left(\alpha|0\rangle + (e^{i\theta})^{-1} e^{i\varphi}\beta|1\rangle\right)$

$= \underbrace{e^{i\theta}}_{\text{global phase}}\left(\alpha|0\rangle + \underbrace{e^{i(\varphi-\theta)}}_{\text{relative phase}}\beta|1\rangle\right)$

$= \alpha|0\rangle + e^{i(\varphi-\theta)}\beta|1\rangle$

probability of measuring 0 $= |\alpha|^2$

probability of measuring 1 $= |e^{i\varphi}\beta|^2 = |e^{i\varphi}|^2|\beta|^2$
$= |\beta|^2$

magnitude of complex number in exponential form $re^{i\varphi}$ is r

Hadamard Gate &

+, -, i, -i states

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
$|i\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$
$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$
$|-i\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

all have equal probability of 0 & 1.
different relative phase

Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$|0\rangle \xrightarrow{H} |+\rangle$ $\qquad$ $|+\rangle \xrightarrow{H} |0\rangle$

$|1\rangle \xrightarrow{H} |-\rangle$ $\qquad$ $|-\rangle \xrightarrow{H} |1\rangle$

Hadamard is its own inverse

$$H(\alpha|0\rangle + e^{i\varphi}\beta|1\rangle)$$

$$= \alpha H|0\rangle + e^{i\varphi}\beta H|1\rangle$$

$$= \alpha|+\rangle + e^{i\varphi}\beta|-\rangle$$

$$= \alpha\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + e^{i\varphi}\beta\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \left(\frac{\alpha + e^{i\varphi}\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha - e^{i\varphi}\beta}{\sqrt{2}}\right)|1\rangle$$

$$\underbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}_{=\,|+\rangle} \xrightarrow{H} |0\rangle$$

$$\underbrace{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle}_{=\,|-\rangle} \xrightarrow{H} |1\rangle$$

$|+\rangle$ & $|-\rangle$ only differ by relative phase.

initially both had the same chance of being measured a 0 & 1.

Phase Gates
(S and T gates)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

$|0\rangle \xrightarrow{S} |0\rangle$

$|1\rangle \xrightarrow{S} e^{i\frac{\pi}{2}}|1\rangle$ adding a relative phase of $e^{i\frac{\pi}{2}}$

$\alpha|0\rangle + \beta|1\rangle \xrightarrow{S} \alpha|0\rangle + e^{i\frac{\pi}{2}}\beta|1\rangle$

$|0\rangle \xrightarrow{T} |0\rangle$

$|1\rangle \xrightarrow{T} e^{i\frac{\pi}{4}}|1\rangle$ adding a relative phase of $e^{i\frac{\pi}{4}}$

$\alpha|0\rangle + \beta|1\rangle \xrightarrow{T} \alpha|0\rangle + e^{i\frac{\pi}{4}}\beta|1\rangle$

$$S^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(-\frac{\pi}{2})} \end{pmatrix} \qquad T^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(-\frac{\pi}{4})} \end{pmatrix}$$

↳ adding a relative phase of $e^{i(-\frac{\pi}{2})}$
= inverse of S gate

↳ adding a relative phase of $e^{i(-\frac{\pi}{4})}$
= inverse of T gate

$S(\alpha|0\rangle + \beta|1\rangle)$

$= \alpha|0\rangle + e^{i\frac{\pi}{2}}\beta|1\rangle$

$S^{\dagger}(\alpha|0\rangle + e^{i\frac{\pi}{2}}\beta|1\rangle)$

$= \alpha|0\rangle + e^{i(-\frac{\pi}{2})}e^{i\frac{\pi}{2}}\beta|1\rangle$

$= \alpha|0\rangle + \beta|1\rangle$

Representing Multiple Qubits Mathematically

Tensor product $\otimes$

$$|0\rangle \otimes |0\rangle = |00\rangle$$

Two qubits in superposition:

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$= \alpha|0\rangle \otimes \gamma|0\rangle + \alpha|0\rangle \otimes \delta|1\rangle + \beta|1\rangle \otimes \gamma|0\rangle + \beta|1\rangle \otimes \delta|1\rangle$$

$$= \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\delta|1\rangle \otimes |1\rangle$$

$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

probability of measuring $|00\rangle = |\alpha\gamma|^2$

" $|01\rangle = |\alpha\delta|^2$

" $|10\rangle = |\beta\gamma|^2$

" $|11\rangle = |\beta\delta|^2$

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}|00\rangle + \frac{1}{\sqrt{2}}\frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}|10\rangle + \frac{1}{\sqrt{2}}\frac{1}{2}|11\rangle$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

$$\left(\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle\right) \otimes \left(\frac{1}{\sqrt{3}}|0\rangle + e^{i\frac{\pi}{4}}\frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}|000\rangle + e^{i\frac{\pi}{4}}\frac{1}{2}|001\rangle + \frac{1}{2\sqrt{6}}|010\rangle + e^{i\frac{\pi}{4}}\frac{1}{2\sqrt{3}}|011\rangle$$
$$+ \frac{1}{2\sqrt{2}}|100\rangle + e^{i\frac{\pi}{4}}\frac{1}{2}|101\rangle + \frac{1}{2\sqrt{6}}|110\rangle + e^{i\frac{\pi}{4}}\frac{1}{2\sqrt{3}}|111\rangle$$

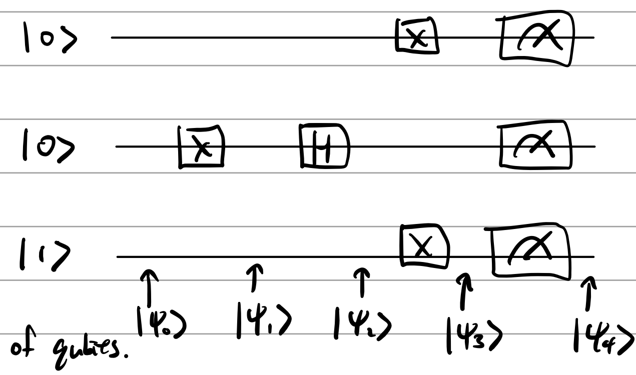
Quantum Circuits

$|000\ldots0\rangle = |0\rangle^{\otimes n}$

n 0's

$|11111\rangle = |1\rangle^{\otimes 5}$

measurements.
measuring the qubit



initial states of qubits.

each line representing a singular qubit

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|\psi_0\rangle = |001\rangle$ $\quad |0\rangle \otimes |0\rangle \otimes |1\rangle$

$|0\rangle \xrightarrow{X} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

$|\psi_1\rangle = |011\rangle$

$|1\rangle \xrightarrow{H} |-\rangle$

$|\psi_2\rangle = |0-1\rangle$

$= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |1\rangle$

$= \frac{1}{\sqrt{2}}(|001\rangle - |011\rangle)$

$|0\rangle \xrightarrow{X} |1\rangle$

$|1\rangle \xrightarrow{X} |0\rangle$

$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|100\rangle - |110\rangle)$

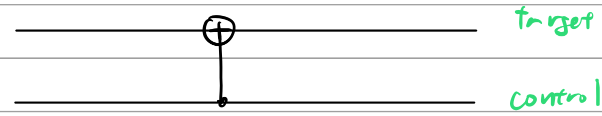
measure

$|\psi_4\rangle$ will be $|100\rangle$ $\frac{1}{2}$ of the time,
$|110\rangle$ $\frac{1}{2}$ of the time.

Multi-qubit gates:

CNOT, Toffoli, and Controlled gates

CNOT / Controlled X gate.



applies X gate to target qubit if control qubit is a 1.

$$CNOT\left(\frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{4}|11\rangle\right)$$

1st qubit is the control. 2nd qubit is the target.

$$= \frac{\sqrt{3}}{4} CNOT|00\rangle + \frac{1}{2} CNOT|01\rangle + \frac{1}{\sqrt{2}} CNOT|\underset{\sim}{1}0\rangle + \frac{1}{4} CNOT|\underset{\sim}{1}1\rangle$$

$$= \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{4}|10\rangle$$

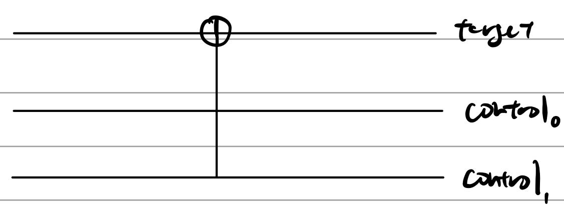
$$CNOT\left(\frac{\sqrt{3}}{2}|001\rangle + \frac{1}{2}|010\rangle\right)$$

$$= \frac{\sqrt{3}}{2} CNOT|001\rangle + \frac{1}{2} CNOT|010\rangle$$

3rd qubit is the control. 2nd qubit is the target.

$$= \frac{\sqrt{3}}{2}|011\rangle + \frac{1}{2}|010\rangle$$

Toffoli gate: Same as CNOT but has 2 control qubits



$$\text{Toffoli}\left(\frac{1}{\sqrt{2}}|0011\rangle + \frac{1}{\sqrt{2}}|0110\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\,\text{Toffoli}\,|0\underset{\sim}{01}1\rangle + \frac{1}{\sqrt{2}}\,\text{Toffoli}\,|0\underset{\sim}{11}0\rangle$$

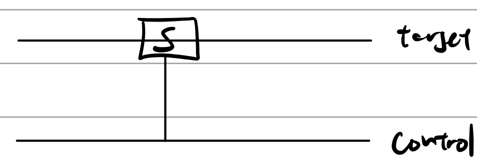
2nd, 3rd qubits are the control. 4th qubit is the target.

$$= \frac{1}{\sqrt{2}}|0011\rangle + \frac{1}{\sqrt{2}}|0111\rangle$$

CNOT gates → create controlled versions of our single qubit gates

Controlled Y = CY

CY, CZ, CS, CT, CH, ...



Measuring Singular Qubits

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{4}|0\underset{\sim}{1}\rangle + \frac{e^{i\frac{\pi}{2}}}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|1\underset{\sim}{1}\rangle$$

probability of measuring 2nd qubit as a 1:

prob (measuring $|01\rangle$) + prob (measuring $|11\rangle$)

$$= \left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2$$

$$= \frac{4}{16} = \frac{1}{4}$$

$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

prob(measuring 1st qubit as a 0)

$= \text{prob(measuring } |00\rangle) + \text{prob(measuring } |01\rangle)$

$= \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2$

$= \frac{1}{2}$

$|\psi_0\rangle = \frac{1}{2}|00\rangle + \frac{1}{4}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$

measure the 1st qubit to be a 1.

$|\psi_1\rangle$ is the state after the measurement

↳ get rid of all the states that don't have 1 as the first qubit.

$$|\psi_1\rangle = \left(\frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle\right) A$$

$$\left(\frac{A}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}A}{4}\right)^2 = 1$$

$$\frac{A^2}{2} + \frac{3A^2}{16} = 1$$

$$\frac{11}{16}A^2 = 1 \qquad A = \frac{4}{\sqrt{11}}$$

$$|\psi_1\rangle = \frac{4}{\sqrt{22}}|10\rangle + \frac{\sqrt{3}}{\sqrt{11}}|11\rangle$$